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Analysis of a time-dependent scheduling problem by signatures of deterioration rate sequences☆

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Abstract

In the paper a single machine time-dependent scheduling problem is considered. The processing time p_j of each job is described by a function of the starting time t of the job, $p_j = 1 + \alpha_j t$, where the job deterioration rate $\alpha_j \geqslant 0$ for $j = 0, 1, \ldots, n$ and $t \geqslant 0$. Jobs are nonpreemptable and independent, there are no ready times and no deadlines. The criterion of optimality of a schedule is the total completion time.

First, the notion of a signature for a given sequence of job deterioration rates is introduced, two types of the signature are defined and their properties are shown. Next, on the basis of these properties a greedy polynomial-time approximation algorithm for the problem is formulated. This algorithm, starting from an initial sequence, iteratively constructs a new sequence concatenating the previous sequence with new elements, according to the sign of one of the signatures of this sequence.

Finally, these results are applied to the problem with job deterioration rates which are consecutive natural numbers, $\alpha_j = j$ for $j = 0, 1, \ldots, n$. Arguments supporting the conjecture that in this case the greedy algorithm is optimal are presented.

Keywords: Single machine scheduling; Deteriorating jobs; Total completion time; Approximate algorithm

1. Introduction

Scheduling jobs with variable processing times is today a self-dependent research field in modern scheduling theory. In problems of this kind, the processing times of jobs are *variable*, contrary to classical scheduling, where they are *fixed*. The variability can be caused by many different factors. For instance, the jobs are executed on machines with varying efficiency or processing times of the jobs depend on a nonrenewable resource.

One of possible approaches to the phenomenon of variability of job processing times is introducing *time-dependent* job processing times: the processing time of a job depends on the starting time of the job, and this dependence is described by a function. In the case when the function is nondecreasing, we deal with *deteriorating* processing times. Note that scheduling with time-dependent processing times of jobs (time-dependent scheduling, in short) can be considered as

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a special case of *discrete–continuous* scheduling, in which also other parameters of the problem (e.g. ready times of jobs, speeds of processors, etc.) are described by functions. We refer the reader to the survey [5] for detailed discussion of these problems.

The practice of everyday life brings us many scheduling problems in which any delay in processing causes an increase of processing times of executed jobs. Since such problems can be modelled as time-dependent scheduling problems, this branch of scheduling has numerous real applications, e.g. in repayment of multiple loans [8], in the modelling of processing jobs on machines with decreasing efficiency [11] or in scheduling maintenance and cleaning assignments [12]. We refer the reader to the review [1] for more details.

The literature of the subject contains about 60 papers published to date. Since in this paper we will consider a time-dependent scheduling problem with linearly deteriorating processing times of jobs and the total completion time criterion, now we will briefly review the known results for linear deterioration and for the above mentioned criterion.

The problem of minimizing the total completion time for a single machine and a set of jobs with proportional processing times, $p_j = \alpha_j t$, where $\alpha_j > 0$, is solvable in $O(n \log n)$ time [12]. The status of the problem with linear processing times, $p_j = \beta_j + \alpha_j t$, where $\alpha_j, \beta_j > 0$, is open even if $\beta_j = 1$ for all jobs. However, in this case for any optimal schedule so-called *V-shape property* holds [11]. This fact allows to decrease the number of candidate optimal schedules from $O(n!)$ to $O(2^n)$. Recently, it has been shown that this property holds also for l_p-norm-related criteria [6].

If we extend the single machine problem by adding deadlines, two results are known. In [4] it is claimed that if arbitrary deadlines are given, a decision version of the problem of minimizing the total completion time for linear processing times is ordinarily NP-complete. A restricted version of the problem, when all deterioration rates are equal, $p_j = \beta_j + \alpha t$, is solvable in $O(n^5)$ time [3]. We refer the reader to the review [1] for more details on a single machine time-dependent scheduling.

Considerably less is known about time-dependent scheduling on parallel machines. The main result obtained so far is that the decision version of the problem of minimizing the total completion time for proportional processing times and two parallel identical machines is ordinarily NP-complete [2,10]. The reader is referred to the survey [4] for more details on parallel machine scheduling with deteriorating jobs.

The paper is organized as follows. In Section 2 we formulate the problem under consideration. In Section 3 we present preliminary results and introduce the notion of signatures of a sequence of job deterioration rates. The basic properties of these signatures and the formulation of a greedy algorithm that finds an approximate solution of the considered problem in polynomial time are presented in Sections 4 and 5, respectively. Section 6 deals with the case when job deterioration rates α_j are consecutive natural numbers. We conjecture that in this case our algorithm is optimal. The paper is completed by conclusions presented in Section 7.

2. Problem formulation

In this paper we consider a single machine scheduling problem with linearly deteriorating processing times and the total completion time criterion. The processing time p_j of each job is described by a function of the starting time t of the job, $p_j = 1 + \alpha_j t$, where *job deterioration rate* $\alpha_j \geqslant 0$ for $j = 0, 1, \ldots, n$ and $t \geqslant 0$. The jobs are nonpreemptable and independent, there are no ready times and no deadlines. The aim is to minimize the total completion time $\sum_{j=0}^{n} C_j$, where

$$C_j = \begin{cases} 1, & j = 0, \\ C_{j-1} + p_j(C_{j-1}) = 1 + (1 + \alpha_j) C_{j-1}, & j = 1, 2, \ldots, n, \end{cases} \tag{1}$$

and the minimization is carried over all permutations of a given sequence $\hat{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ of job deterioration rates. The problem was formulated for the first time in [11] and its time complexity is still unknown. Throughout the paper, for simplicity of presentation, we will call the problem the Time-Dependent Minimum Total Completion Time (TDMTCT) problfem.

In the paper we present a polynomial-time approximate algorithm for the TDMTCT problem, using an approach based on the behaviour of *signatures* of a sequence of job deterioration rates.

3. Preliminaries

Note that formula (1) can be rewritten for $j = 0, 1, \dots, n$ as a superposition

$$C_j = f_j(f_{j-1}(\cdots(f_0(t))\cdots)) \tag{2}$$

of functions $f_j(t) = t + p_j(t) = 1 + a_j t$, where $a_j = 1 + \alpha_j$ for $j = 0, 1, \dots, n$ and job 0 starts at time $t = 0$.

Expanding Eq. (2) for $j = 0, 1, \dots, n$, we have $C_0 = 1,\ C_1 = a_1 C_0 + 1,\ C_2 = a_2 C_1 + 1, \dots, C_n = a_n C_{n-1} + 1$. This set of linear equations, in turn, can be rewritten in the matrix form as follows:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -a_1 & 1 & \cdots & 0 & 0 \\ 0 & -a_2 & \cdots & 0 & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & -a_n & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{3}$$

Eq. (3) will be denoted by $A(a)C(a) = d(1)$, where $A(a)$ is the matrix given above, $C(a) = [C_0, C_1, \dots, C_n]^\top \in \mathbb{R}^{n+1}$ and $d(1) = [1, 1, \dots, 1]^\top \in \mathbb{R}^{n+1}$.

Remark 1. It is easy to see that if the starting time of the first job is $t = 0$, the coefficient a_0 has no effect on the value of C_j, for $j=0, 1, \dots, n$. Moreover, C_j depends on a_i in a monotone nondecreasing way for each $i=1, 2, \dots, j$. Therefore, given any ordering of the sequence $\hat{a}$, the best strategy is to set as a_0 the maximal element in this sequence [11]. In other words, if we start at $t = 0$, the subject of our interest is the remaining n-element subsequence $a = (a_1, a_2, \dots, a_n)$ of the sequence $\hat{a}$, with a_0 maximal. Hence, from now on, we will assume that a_0 is the maximal element in $\hat{a}$ and we will consider only the vector a.

Throughout the paper we will denote by Π_n, $\pi \in \Pi_n$ and a_π the set of all permutations of the sequence $(1, 2, \dots, n)$, arbitrary permutation from Π_n and the sequence a ordered according to π, respectively. Under this notation, the TDMTCT problem is equivalent to finding a permutation $\pi^* \in \Pi_n$ such that the total completion time for this sequence, $\sum_{j=1}^{n} C_j(a_{\pi^*})$, is minimal, i.e.

$$\sum_{j=1}^{n} C_j(a_{\pi^*}) = \min \left\{ \sum_{i=1}^{n} C_i(a_\pi) : \pi \in \Pi_n \right\},$$

where $C(a_\pi)$ is the solution of Eq. (3) for the sequence a_π.

The determinant $\det(A(a)) = 1$ and hence for the matrix $A(a)$ there exists the inverse matrix $A^{-1}(a)$. This inverse matrix has the following form:

$$A^{-1}(a) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_1 & 1 & \cdots & 0 & 0 \\ a_1 a_2 & a_2 & \cdots & 0 & 0 \\ a_1 a_2 a_3 & a_2 a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_1 a_2 \cdots a_n & a_2 a_3 \cdots a_n & \cdots & a_n & 1 \end{bmatrix}. \tag{4}$$

By (4), the components of the solution $C(a) = A^{-1}(a)d(1)$ can be found, i.e.

$$C_j(a) = \sum_{i=1}^{j+1} \prod_{k=i}^{j} a_k$$

for $j = 0, 1, \dots, n$, where an empty product is assumed to be equal to 1.

Similarly, if $\bar{a} = (a_n, a_{n-1}, \ldots, a_1)$ denotes the reverse permutation of elements of the sequence a, then $C(\bar{a}) = A^{-1}(\bar{a})d(1)$ and

$$C_j(\bar{a}) = \sum_{i=n-j}^{n} \prod_{k=n-j+1}^{i} a_k$$

for $j = 0, 1, \ldots, n$.

We will apply two l_p-norm-related measures (see, e.g., [9] for definition of l_p-norm and its properties) to the vector $C(a)$, namely

$$F(a) = \sum_{j=1}^{n} \sum_{i=1}^{j} \prod_{k=i}^{j} a_k \tag{5}$$

and

$$M(a) = \sum_{i=1}^{n+1} \prod_{k=i}^{n} a_k. \tag{6}$$

Note that since there hold

$$\|C(a)\|_1 = \sum_{j=0}^{n} C_j(a) = \sum_{j=0}^{n} \sum_{i=1}^{j+1} \prod_{k=i}^{j} a_k = F(a) + (n+1)$$

and

$$\|C(a)\|_\infty = \max_{0 \leqslant j \leqslant n} \{C_j\} = M(a),$$

it implies that $\|C(a)\|_1 \equiv \sum C_j$ and $\|C(a)\|_\infty \equiv C_{\max}$, see [6] for details.

We will refer to minimizing of $F(a)$ as to the $F(a)$-*problem*. Recall that if $a = a_{\pi^*}$ is the optimal solution of the $F(a)$-problem, then $\bar{a} = (a_n, a_{n-1}, \ldots, a_1)$ is also optimal solution of the $F(a)$-problem [11]. Moreover, for this $a = a_{\pi^*}$ the *V-shape property* must hold [6,11]. (This means that the sequence is nonincreasing for $i = 1, 2, \ldots, m$, and is nondecreasing for $i = m, m+1, \ldots, n$, where $1 < m < n$. We will call such sequences *V-sequences*.)

Similarly, the minimization of $M(a)$ we will call the $M(a)$-*problem*. It is known (see, e.g., [7,13]) that the solution of the $M(a)$-problem is a nonincreasing sequence. (It is worth to note that $M(a) = C_n(a)$ and $M(\bar{a}) = C_n(\bar{a})$. Moreover, $M(a)$ and $M(\bar{a})$ are nothing else than the sums of the elements in the last row and in the first column of matrix (4), respectively.)

For a given sequence $a = (a_1, a_2, \ldots, a_n)$, define two *signatures* of a as follows:

$$S^-(a) \equiv M(\bar{a}) - M(a) = \sum_{i=1}^{n} \prod_{j=1}^{i} a_j - \sum_{i=1}^{n} \prod_{j=i}^{n} a_j \tag{7}$$

and

$$S^+(a) \equiv M(\bar{a}) + M(a). \tag{8}$$

Signatures (7) and (8) reflect some intrinsic properties as of the matrix $A(a)$ as of $F(a)$- and $M(a)$-problems, and play crucial role in our further considerations. Hence we will prove now some of their properties.

4. Basic properties of signatures

Introduce the following notation. Given a sequence $a = (a_1, a_2, \ldots, a_n)$ and any two numbers $\alpha \geqslant 1$ and $\beta \geqslant 1$, let $(\alpha|a|\beta)$ and $(\beta|a|\alpha)$ denote concatenations of α, β and a in the indicated orders, respectively. (Sometimes it will be more convenient to write a_0 and a_{n+1} instead of α and β, respectively.)

Presented further results show relations which hold between the values of functionals $F(\cdot)$ and $M(\cdot)$ and the values of signatures $S^-(\cdot)$ and $S^+(\cdot)$, calculated for a given sequence a and the sequence obtained by adding at the beginning and the end of a two new elements.

For simplicity of notation, let $\mathscr{A} = \prod_{j=1}^{n} a_j$.

Lemma 2. *For a given sequence a and any numbers* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$, *there hold the following equalities*:

$$F(\alpha|a|\beta) = F(a) + \alpha M(\bar{a}) + \beta M(a) + \alpha \cdot \mathscr{A} \cdot \beta \tag{9}$$

and

$$F(\beta|a|\alpha) = F(a) + \beta M(\bar{a}) + \alpha M(a) + \alpha \cdot \mathscr{A} \cdot \beta. \tag{10}$$

Proof. Let $a = (a_1, a_2, \ldots, a_n)$, $a_0 = \alpha \geqslant 1$, $a_{n+1} = \beta \geqslant 1$. Then we have

$$\begin{aligned} F(a_0|a|a_{n+1}) &= \sum_{j=0}^{n+1} \sum_{i=0}^{j} a_i a_{i+1} \cdots a_j \\ &= F(a) + \sum_{j=0}^{n} a_0 a_1 \cdots a_j + \sum_{i=0}^{n+1} a_i a_{i+1} \cdots a_{n+1} \\ &= F(a) + a_0 \sum_{j=0}^{n} a_1 a_2 \cdots a_j + a_{n+1} \sum_{i=1}^{n+1} a_i a_{i+1} \cdots a_n + a_0 a_1 \cdots a_n a_{n+1}, \end{aligned}$$

and equality (9) follows. To get equality (10), it is sufficient to exchange α and β and to note that the last term remains unchanged during this exchange. □

The above lemma, by formulae (9) and (10), shows how to calculate the values of the functional $F(\cdot)$ for the sequence a extended by the elements α and β, if we know the values of $F(a)$, $M(a)$ and $M(\bar{a})$.

By Lemma 2 and definitions (7) and (8), we obtain general formulae concerning the difference and the sum of values of $F(\cdot)$ for sequences $(\alpha|a|\beta)$ and $(\beta|a|\alpha)$.

Lemma 3. *For a given sequence a and any numbers* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$, *there hold the following equalities*:

$$F(\alpha|a|\beta) - F(\beta|a|\alpha) = (\alpha - \beta) S^-(a) \tag{11}$$

and

$$F(\alpha|a|\beta) + F(\beta|a|\alpha) = (\alpha + \beta) S^+(a) + 2(F(a) + \alpha \cdot \mathscr{A} \cdot \beta). \tag{12}$$

Proof. Let there $a = (a_1, a_2, \ldots, a_n)$, $\alpha \geqslant 1$ and $\beta \geqslant 1$ be given. Then by subtracting the left and the right sides of equalities (9) and (10), respectively, and by applying definition (7) of $S^-(a)$ signature, Eq. (11) follows.

Similarly, by adding the left and the right sides of equalities (9) and (10), respectively, and by applying definition (8) of $S^+(a)$ signature, we obtain Eq. (12). □

From identities (11) and (12) we obtain yet another pair of identities (which are equivalent to these in Lemma 2) expressed in terms of signatures $S^-(\cdot)$ and $S^+(\cdot)$.

Lemma 4. *For a given sequence a and any numbers* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$, *there hold the following equalities*:

$$F(\alpha|a|\beta) = F(a) + \tfrac{1}{2}((\alpha + \beta) S^+(a) + (\alpha - \beta) S^-(a)) + \alpha \cdot \mathscr{A} \cdot \beta \tag{13}$$

and

$$F(\beta|a|\alpha) = F(a) + \tfrac{1}{2}((\alpha + \beta) S^+(a) - (\alpha - \beta) S^-(a)) + \alpha \cdot \mathscr{A} \cdot \beta. \tag{14}$$

Proof. Indeed, by adding the left and the right sides of equalities (11) and (12), respectively, we obtain equality (13). Similarly, by subtracting the left and the right sides of equalities (12) and (11), respectively, we obtain equality (14). □

The next result shows how to concatenate new elements α and β with the given sequence a in order to decrease the value of the functional $F(\cdot)$.

Theorem 5. *Let there be given the sequence a related to the $F(a)$-problem and the numbers $\alpha \geqslant 1$ and $\beta \geqslant 1$. Then there holds the following equivalence*:

$$F(\alpha|a|\beta) \leqslant F(\beta|a|\alpha) \quad \textit{iff} \quad (\alpha - \beta)S^-(a) \leqslant 0. \tag{15}$$

Moreover, there holds the similar equivalence, if in equivalence (15) *we replace the symbol* '$\leqslant$' by '$\geqslant$', *respectively*.

Proof. The result is a consequence of identity (11) in Lemma 3. □

From Theorem 5 it follows that in order to decrease the value of $F(\cdot|a|\cdot)$ we should choose $a^{\mathrm{L}} = (\alpha|a|\beta)$ instead of $a^{\mathrm{R}} = (\beta|a|\alpha)$ if $(\alpha - \beta)S^-(a) \leqslant 0$, and a^{R} instead of a^{L} in the opposite case. Thus the behaviour of the functional $F(\cdot)$ during such concatenations is controlled by the sign of the signature $S^-(a)$ of the original sequence a.

In the next theorem we give a greedy strategy for solving the $F(a)$-problem. This strategy is based on behaviour of the signature $S^-(\cdot)$ only.

Theorem 6. *Let there a be a nondecreasingly ordered sequence for the $F(a)$-problem, $u = (u_1, u_2, \ldots, u_{k-1})$ be a V-sequence constructed from the first $k-1$ elements of a, $\alpha = a_k \geqslant 1$, $\beta = a_{k+1} \geqslant 1$, where $1 < k < n$ and $\alpha \leqslant \beta$. Then there holds the following implication*:

$$\textit{if } S^-(u) \geqslant 0 \quad \textit{then} \quad F(\alpha|u|\beta) \leqslant F(\beta|u|\alpha). \tag{16}$$

Moreover, there holds the similar implication, if in implication (16) *we replace the symbol* '$\geqslant$' *by* '$\leqslant$' *and the symbol* '$\leqslant$' *by* '$\geqslant$', *respectively*.

Proof. Assume that the sign of the signature $S^-(u)$ is known. Then it is sufficient to note that knowing the sign of the difference $\alpha - \beta$ we know, by equivalence (15), the sign of the difference $F(\alpha|u|\beta) - F(\beta|u|\alpha)$. □

The next result shows a relation between signatures of sequences $(\alpha|a|\beta)$ and $(\beta|a|\alpha)$ and values of the functional $M(\cdot)$ for sequences a and $\bar{a}$.

Theorem 7. *For a given sequence a and any numbers $\alpha \geqslant 1$ and $\beta \geqslant 1$, there hold the following equalities*:

$$S^-(\alpha|a|\beta) = \alpha M(\bar{a}) - \beta M(a) \tag{17}$$

and

$$S^-(\beta|a|\alpha) = \beta M(\bar{a}) - \alpha M(a). \tag{18}$$

Proof. Let $a = (a_1, a_2, \ldots, a_n)$, $a_0 = \alpha \geqslant 1$ and $a_{n+1} = \beta \geqslant 1$. Then we have

$$\begin{aligned} S^-(\alpha|a|\beta) &= \sum_{i=0}^{n+1} a_0 a_1 \cdots a_i - \sum_{i=0}^{n+1} a_i \cdots a_n a_{n+1} \\ &= a_0 \sum_{i=1}^{n} a_1 a_2 \cdots a_i - a_{n+1} \sum_{i=1}^{n} a_i a_{i+1} \cdots a_n \\ &= a_0 M(\bar{a}) - a_{n+1} M(a), \end{aligned}$$

and identity (17) follows. In similar way, by exchanging α and β and considering $\bar{a}$ instead of a, we obtain identity (18). □

As a corollary from Theorem 7 we obtain identities which explain the behaviour of consecutively calculated signatures $S^{-}(\cdot)$.

Theorem 8. *For a given sequence a and any numbers* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$, *there hold the following identities*:

$$S^{-}(\alpha|a|\beta) + S^{-}(\beta|a|\alpha) = (\alpha+\beta)S^{-}(a), \tag{19}$$

$$S^{-}(\alpha|a|\beta) - S^{-}(\beta|a|\alpha) = (\alpha-\beta)S^{+}(a) \tag{20}$$

and

$$\begin{aligned} S^{-}(\alpha|a|\beta)^2 - S^{-}(\beta|a|\alpha)^2 &= (\alpha^2-\beta^2)(M(\bar{a})^2 - M(a)^2) \\ &= (\alpha^2-\beta^2)S^{-}(a)S^{+}(a). \end{aligned} \tag{21}$$

Proof. Indeed, by adding the left and the right sides of equalities (17) and (18), respectively, we obtain identity (19). Similarly, by subtracting the left and the right sides of equalities (17) and (18), respectively, we obtain identity (20).

Multiplying the left and the right sides of identities (19) and (20), respectively, we obtain identity (21). □

Remark 9. The analysis of these identities shows that in general we cannot determine uniquely the sign of signatures $S^{-}(\alpha|a|\beta)$ and $S^{-}(\beta|a|\alpha)$ in terms of the sign of the signature $S^{-}(a)$ only, even if we know that $\alpha \leqslant \beta$ (or $\alpha \geqslant \beta$). Indeed, if we know the sign of $F(\alpha|a|\beta) - F(\beta|a|\alpha)$ or, equivalently, the sign of $(\alpha-\beta)S^{-}(a)$, then from identities (19), (20) and (21) it follows that for the consecutive signatures we only know the sign of $|S^{-}(\alpha|a|\beta)| - |S^{-}(\beta|a|\alpha)|$.

Finally, note that by Theorem 8 we can prove yet another pair of results.

Lemma 10. *For a given sequence a and any numbers* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$, *there hold the following identities*:

$$S^{-}(\alpha|a|\beta) = \tfrac{1}{2}((\alpha+\beta)S^{-}(a) + (\alpha-\beta)S^{+}(a)) \tag{22}$$

and

$$S^{-}(\beta|a|\alpha) = \tfrac{1}{2}((\alpha+\beta)S^{-}(a) - (\alpha-\beta)S^{+}(a)). \tag{23}$$

Proof. Indeed, by adding the left and the right sides of identities (19) and (20), respectively, we obtain identity (22). Similarly, by subtracting the left and the right sides of identities (20) and (19), respectively, we obtain identity (23). □

Remark 11. Considering the sequence $\bar{a}$ instead of a, one can formulate and prove counterparts of all lemmas and theorems from this section. We omit them, since they do not introduce new insights into the problem. Note only that there holds the equality

$$S^{-}(a) + S^{-}(\bar{a}) = 0,$$

i.e. the signatures $S^{-}(a)$ and $S^{-}(\bar{a})$ have opposite signs.

5. Greedy algorithm

In this section we will introduce a new algorithm for the TDMTCT problem, based on properties of signatures. However, we start with a presentation of two approximate algorithms for this problem, introduced in [11].

5.1. Mosheiov's algorithms

In [11] two heuristics for the considered problem have been proposed, both running in $O(n \log n)$ time. The first of them, H_1, starts with ordering the jobs nonincreasingly with respect to job deterioration rates. Then it creates a V-shape by alternately assigning unscheduled yet jobs to the left and to the right branch of the constructed V-sequence.

Let "Section 1" be the set of jobs scheduled before the job with minimal deterioration rate (the left branch of a V-sequence) and "Section 2" be the set of remaining jobs (the right branch of a V-sequence).

Algorithm H_1 for the TDMTCT problem
Step 1. { `Initialization` }
Arrange the jobs in descending order of deterioration rates. Call this descending sequence DS.
Step 2. { `Main loop` }
Assign the first job of DS to be scheduled first, the second job of DS to be scheduled last, the third job of DS to be scheduled second, and the fourth job of DS to be scheduled next to last. Continue by adding jobs alternately to each section (add to the first available place in Section 1 and to the last available place in Section 2).

The second algorithm, H_2, creates a V-shape by assigning jobs to the left (the right) branch of the constructed V-sequence, if the sum of deterioration rates for the left (the right) branch is not greater (is greater) than this sum for the right (the left) branch. As previously, the algorithm starts with the input jobs sequence ordered nonincreasingly with respect to job deterioration rates.

Let "Section 1" and "Section 2" be defined as previously and let SUM1 (SUM2) be the current sum of deterioration rates of jobs in Section 1 (Section 2).

Algorithm H_2 for the TDMTCT problem
Step 1. { `Initialization` }
Arrange the jobs in descending order of deterioration rates. Call this descending sequence DS.
Step 2. { `Main loop` }
Assign the first job of DS to be scheduled first, the second job of DS to be scheduled last, and the third job of DS to be scheduled next to last. Continue by adding at each step the next job of DS to Section 1 (in the first available place) if SUM1 $\leqslant$ SUM2, and to Section 2 (in the last available position) if SUM2 $<$ SUM1.

5.2. Greedy algorithm

Let u denote a V-shaped sequence composed from the first $k \geqslant 1$ elements of ordered nondecreasingly sequence a, with the actual value of the functional $F(\cdot)$ equal to $F(u)$. Let $\alpha = a_{k+1} \geqslant 1$ and $\beta = a_{k+2} \geqslant 1$ be two consecutive elements of a. Clearly, $\alpha \leqslant \beta$. Then letting $u^{\mathrm{L}} = (\alpha|u|\beta)$ and $u^{\mathrm{R}} = (\beta|u|\alpha)$ we obtain two possibilities to extend simultaneously the left branch and the right branch of the sequence u by the concatenation α on the left and β on the right, or conversely. On the basis of results from Section 4, we can formulate the following algorithm.

Greedy algorithm *GA* for the TDMTCT problem
Input: the sequence $\hat{a} = (a_0, a_1, \ldots, a_n)$
Output: the suboptimal V-sequence $u = (a_0, u_1, \ldots, u_n)$

Step 1. { `Initialization` }
Sort the sequence $\hat{a}$ in a nondecreasing order, $a_{[1]} \leqslant a_{[2]} \leqslant \cdots \leqslant a_{[n]} \leqslant a_0$;
Step 2. { `Main loop` }
If n is odd **then**
begin
$u := (a_{[1]})$;
for $i := 2$ **to** $n-1$ **step** 2 **do**
if $S^-(u) \leqslant 0$ **then** $u := (a_{[i+1]}|u|a_{[i]})$ **else** $u := (a_{[i]}|u|a_{[i+1]})$
end
else { n `is even` }
begin
$u := (a_{[1]}, a_{[2]})$;
for $i := 3$ **to** $n-1$ **step** 2 **do**
if $S^-(u) \leqslant 0$ **then** $u := (a_{[i+1]}|u|a_{[i]})$ **else** $u := (a_{[i]}|u|a_{[i+1]})$
end
Step 3. { `Final sequence` }
$u := (a_0|u)$.

The running time of the algorithm is $O(n \log n)$.

Table 1
Experiment results for the sequences $a = (1, 2, \ldots, n)$

n	$OPT(I)$	$R_{GA}(I)$	$R_{H_1}(I)$	$R_{H_2}(I)$
2	8	⋆	⋆	⋆
3	21	⋆	⋆	0.142857142857
4	65	⋆	0.015384615385	0.138461538462
5	250	⋆	0.008000000000	0.084000000000
6	1232	⋆	0.008928571429	0.060876623377
7	7559	⋆	0.003571901045	0.053049345151
8	55,689	⋆	0.002621702670	0.033884609169
9	475,330	⋆	0.000995098142	0.020871815370
10	4,584,532	⋆	0.000558835667	0.014906428835
11	49,111,539	⋆	0.000244423210	0.011506155407
12	577,378,569	⋆	0.000142137247	0.009070282967
13	7,382,862,790	⋆	0.000080251254	0.007401067385
14	101,953,106,744	⋆	0.000052563705	0.006210868342
15	1,511,668,564,323	⋆	0.000035847160	0.005304460215
16	23,947,091,701,857	⋆	0.000025936659	0.004588979235
17	403,593,335,602,130	⋆	0.000019321905	0.004013033262
18	7,209,716,105,574,116	⋆	0.000014779355	0.003541270022
19	136,066,770,200,782,755	⋆	0.000011522779	0.003149229584
20	2,705,070,075,537,727,250	⋆	0.000009131461	0.002819574105

Example 12. Let a=(2, 3, 4, 6, 8, 16, 21). The optimal V-shaped sequence is $a^\star$=(21, 8, 6, 3, 2, 4, 16), with $\sum C_j(a^\star)$=23, 226. The algorithm *GA* generates the V-shaped sequence $u = (21, 8, 6, 2, 3, 4, 16)$, with $\sum C_j(u) =$ 23, 240.

The algorithms H_1 and H_2 give worse results: $a_{H_1} = (21, 8, 4, 2, 3, 6, 16)$, $a_{H_2} = (21, 6, 3, 2, 4, 8, 16)$, with $\sum C_j(a_{H_1}) = 23, 418$ and $\sum C_j(a_{H_2}) = 24, 890$, respectively.

Thus, in general, the algorithm *GA* is not optimal. We conjecture, however, that this algorithm is optimal for sequences of consecutive natural numbers.

Example 13. Let $a = (2, 3, 4, 5, 6, 7, 8)$. The algorithm *GA* generates the optimal V-sequence $a^\star = (8, 6, 5, 2, 3, 4, 7)$, with $\sum C_j(a^\star) = 7386$. The solutions obtained by algorithms H_1 and H_2 are as follows: $a_{H_1} = (8, 6, 4, 2, 3, 5, 7)$ and $a_{H_2} = (8, 5, 4, 2, 3, 6, 7)$, with $\sum C_j(a_{H_1}) = 7403$ and $\sum C_j(a_{H_2}) = 7638$, respectively.

In order to compare the greedy algorithm *GA* with algorithms H_1 and H_2 we have conducted a computational experiment. As a measure of the quality of a schedule generated by an algorithm A we used the ratio

$$R_A(I) = \frac{A(I) - OPT(I)}{OPT(I)},$$

where $A(I)$ and $OPT(I)$ denote, for a given instance I, the value of the total completion time obtained by the algorithm A and the optimal total completion time, respectively. In our experiment job deterioration rates were consecutive natural numbers, $\alpha_j = j$ for $j = 0, 1, \ldots, n$. The results of this experiment are summarized in Table 1. (The star ⋆ denotes the fact that for particular values of n and I the ratio $R_A(I)$ is equal to 0.)

We conjecture also that in the case of any sequence of consecutive natural numbers one can construct an optimal schedule knowing only the form of the sequence a, *without* calculation of the signature $S^-(u)$. Arguments supporting this conjecture are given in the next section.

6. Signatures of sequences of consecutive natural numbers

We begin this section with a formula that can be derived from definition (7) of the signature $S^-(a)$. For simplicity of notation, if the sequence a is fixed, we will write S_n^- instead of $S^-(a)$:

$$S_n^- = \sum_{i=1}^{m} a_1 a_2 \cdots a_i - \sum_{i=1}^{n-m} a_{n-i+1} a_{n-i+2} \cdots a_n + \sum_{i=1}^{n-m} a_1 a_2 \cdots a_{m+i} - \sum_{i=1}^{m} a_i a_{i+1} \cdots a_n,$$

where $1 \leqslant m \leqslant n$.

From this formula one can obtain the following useful representation of the signature in the case $n = 2m$ and $n = 2m - 1$, respectively.

Lemma 14. *Let $a = (a_1, a_2, \ldots, a_n)$. If $n = 2m$, then*

$$S_{2m}^{-} = \sum_{i=1}^{m} \eta_i(m) \left(\prod_{j=1}^{m-i+1} a_j - \prod_{j=m+i}^{2m} a_j \right), \tag{24}$$

where $\eta_1(m) = 1$ and $\eta_i(m) = 1 + \prod_{j=m-i+2}^{m+i-1} a_j$ for $i = 2, 3, \ldots, m$. If $n = 2m - 1$, then

$$S_{2m-1}^{-} = \sum_{i=1}^{m-1} \omega_i(m) \left(\prod_{j=1}^{m-i} a_j - \prod_{j=m+i}^{2m-1} a_j \right), \tag{25}$$

where $\omega_i(m) = 1 + \prod_{j=m-i+1}^{m+i-1} a_j$ for $i = 1, 2, \ldots, m - 1$.

Proof. Let $n = 2m$. Then

$$S_{2m}^{-} = \sum_{i=1}^{m} (a_1 a_2 \cdots a_i - a_{2m-i+1} a_{2m-i+2} \cdots a_{2m}) + \sum_{i=1}^{m} a_1 a_2 \cdots a_{m+i} - \sum_{i=1}^{m} a_i a_{i+1} \cdots a_{2m}.$$

Reducing the last term in the second sum with the first one in the third sum we obtain

$$\begin{aligned} S_{2m}^{-} = & \sum_{i=1}^{m} (a_1 a_2 \cdots a_i - a_{2m-i+1} a_{2m-i+2} \cdots a_{2m}) \\ & + \sum_{i=1}^{m-1} a_{i+1} a_{i+2} \cdots a_{2m-i} \cdot (a_1 a_2 \cdots a_i - a_{2m-i+1} a_{2m-i+2} \cdots a_{2m}). \end{aligned}$$

Joining the both sums and changing the index of summation according to $i := m - i + 1$, we obtain

$$\begin{aligned} S_{2m}^{-} = & \sum_{i=2}^{m} (1 + a_{m-i+2} a_{m-i+3} \cdots a_{m+i-1}) \cdot (a_1 a_2 \cdots a_{m-i+1} - a_{m+i} a_{m+i+1} \cdots a_{2m}) \\ & + (a_1 a_2 \cdots a_m - a_{m+1} a_{m+2} \cdots a_{2m}). \end{aligned}$$

Thus, taking into account definitions of the coefficients η_i, formula (24) follows.

To prove formula (25) we proceed in the same way. Let $n = 2m - 1$. Then

$$S_{2m-1}^{-} = \sum_{i=1}^{m} a_1 a_2 \cdots a_i - \sum_{i=1}^{m-1} a_{m+i} a_{m+i+1} \cdots a_{2m-1} + \sum_{i=1}^{m-1} a_1 a_2 \cdots a_{m+i} - \sum_{i=1}^{m} a_i a_{i+1} \cdots a_{2m-1}.$$

Changing the index of the summation in the first sum according to $i := m - i$ and then in the third sum according to $i := i - 1$ and in the last sum according to $i := m - i + 1$ we obtain

$$\begin{aligned} S_{2m-1}^{-} = & \sum_{i=0}^{m-1} a_1 a_2 \cdots a_{m-i} - \sum_{i=1}^{m-1} a_{m+i} a_{m+i+1} \cdots a_{2m-1} \\ & + \sum_{i=2}^{m-1} (a_1 a_2 \cdots a_{m-i}) \cdot (a_{m-i+1} a_{m-i+2} \cdots a_{m+i-1}) \\ & - \sum_{i=1}^{m-1} (a_{m-i+1} a_{m-i+2} \cdots a_{m+i-1}) \cdot (a_{m+i} a_{m+i+1} \cdots a_{2m-1}). \end{aligned}$$

Moving in the first sum the term with the index $i=0$ to the third one under the index $i=1$ and applying definitions of the coefficients ω_i, the proof of formula (25) is completed. $\square$

Conducted computational experiment has shown that optimal solutions of the $F(a)$-problem, where $a=(1, 2, \ldots, n)$ and $n=2, 3, \ldots, 20$, are given by the following ordering of elements of the vector a:

$$\begin{array}{lcccccccccccccccccc}
m=1 & & & & 1 & 2 & & & & & & & 3 & 1 & 2 & & & \\
m=2 & & & 4 & 1 & 2 & 3 & & & & & 4 & 3 & 1 & 2 & 5 & & \\
m=3 & & 5 & 4 & 1 & 2 & 3 & 6 & & & 7 & 4 & 3 & 1 & 2 & 5 & 6 & \\
m=4 & 8 & 5 & 4 & 1 & 2 & 3 & 6 & 7 & 8 & 7 & 4 & 3 & 1 & 2 & 5 & 6 & 9 \\
\ldots & & & & & \ldots & & & & & & & & \ldots & & & &
\end{array}$$

where the left sequences correspond to the case $n=2m$ and the right sequences correspond to $n=2m+1$. The same ordering of elements of the vector a is generated at the output of the greedy algorithm *GA* for $n=2, 3, \ldots, 20$.

Consider sequences a related to $F(a)$-problem by the formulae

$$a=(r_m+(-1)^m, \ldots, r_2+1, r_1-1, r_1, r_2, \ldots, r_m) \quad \text{for } n=2m, \tag{26}$$

and

$$a=(s_{m-1}+2, \ldots, s_2+2, s_1+2, s_1, s_2, \ldots, s_m) \quad \text{for } n=2m-1, \tag{27}$$

where

$$r_k=2k-\tfrac{1}{2}((-1)^k+3)+1, \quad k=1, 2, \ldots, m \quad \text{for } n=2m, \tag{28}$$

and

$$s_k=2k-\tfrac{1}{2}((-1)^k+3), \quad k=1, 2, \ldots, m \quad \text{for } n=2m-1. \tag{29}$$

We will refer to r_k and s_k sequences and to the related sequence a as to the *even* and the *odd* sequence, respectively. (Note that sequences given immediately after the proof of Lemma 14 are computed using sequences r_k and s_k, respectively.)

Lemma 15. *Let $n=2m$ and a be the even sequence. Then for each integer $m \geqslant 1$ there holds the following equality*:

$$S_{2m}^{-}=\sum_{i=1}^{m}\eta_i\left(\prod_{j=i}^{m}(r_j+(-1)^j)-\prod_{j=i}^{m}r_j\right), \tag{30}$$

where

$$\eta_1=1 \text{ and } \eta_i=1+\prod_{j=1}^{i-1}r_j\prod_{j=1}^{i-1}(r_j+(-1)^j) \tag{31}$$

for $i=2, 3, \ldots, m$.

Proof. Let a be the even sequence and let $n=2m$. Applying Lemma 14 for the sequence a given by formulae (26) and (28), formula (30) for the signature S_{2m}^{-} is obtained. $\square$

Now, on the basis of (30), we can state the first main result in this section.

Theorem 16. *Let $n=2m$ and a be the even sequence. Then for the signatures S_{2m+2}^{-} and S_{2m}^{-} there holds the following formula*:

$$S_{2m+2}^{-}=r_{m+1}S_{2m}^{-}+(-1)^{m+1}\sum_{i=1}^{m+1}\eta_i\prod_{j=i}^{m}(r_j+(-1)^j), \tag{32}$$

where the signature S^-_{2m} *and coefficients* η_i *are defined by formulae* (30) *and* (31), *respectively. Moreover*, *there holds the following identity*:

$$S^-_{2m+2} = R_m((-1)^{m+1} + \Theta_m), \tag{33}$$

where $\Theta_m = (S^-_{2m}/R_m)(r_{m+1} + (-1)^{m+1})$ *and* $R_m = \sum_{i=1}^{m+1} \eta_i \prod_{j=i}^{m} r_j$.

Proof. Let a be the even sequence and let $n = 2m$. Applying Lemma 15 we obtain

$$\begin{aligned}
S^-_{2m+2} &= \sum_{i=1}^{m+1} \eta_i((r_i + (-1)^i) \cdots (r_{m+1} + (-1)^{m+1}) - r_i \cdots r_{m+1}) \\
&= \sum_{i=1}^{m} \eta_i((r_i + (-1)^i) \cdots (r_{m+1} + (-1)^{m+1}) - r_i \cdots r_{m+1}) \\
&\quad + \eta_{m+1}((r_{m+1} + (-1)^{m+1}) - r_{m+1}) \\
&= r_{m+1} S^-_{2m} + (-1)^{m+1} \sum_{i=1}^{m} \eta_i((r_i + (-1)^i) \cdots (r_m + (-1)^m)) \\
&\quad + \eta_{m+1}((r_{m+1} + (-1)^{m+1}) - r_{m+1}) \\
&= r_{m+1} S^-_{2m} + (-1)^{m+1} \sum_{i=1}^{m+1} \eta_i((r_i + (-1)^i) \cdots (r_m + (-1)^m)),
\end{aligned}$$

and formula (32) follows in view of definition of coefficients η_i. Formula (33) follows immediately from assumed notation and formula (32). $\square$

Now consider the case of the odd sequence. Applying Lemma 14 for the sequence a given by formulae (27) and (29), we obtain the following formula for the signature S^-_{2m-1}.

Lemma 17. *Let* $n = 2m - 1$ *and a be the odd sequence. Then for each integer* $m \geqslant 1$ *there holds the following equality*:

$$S^-_{2m-1} = \sum_{i=1}^{m-1} \omega_i \left(\prod_{j=i}^{m-1} (s_j + 2) - \prod_{j=i+1}^{m} s_j \right), \tag{34}$$

where $\omega_i = 1 + \prod_{j=1}^{i} s_j \prod_{j=1}^{i-1} (s_j + 2)$ *for* $i = 1, 2, \ldots, m - 1$.

Proof. Let a be the odd sequence and let $n = 2m - 1$. Then we have $a_{m-i} = s_i + 2$ for $i = 1, 2, \ldots, m - 1$ and $a_{m+i-1} = s_i$ for $i = 1, 2, \ldots, m$. Substituting these values in formula (25) and noticing that

$$\omega_i = 1 + a_{m-i+1} \cdots a_{m+i-1} = 1 + s_1 \cdots s_i (s_1 + 2) \cdots (s_{i-1} + 2),$$

formula (34) follows. $\square$

On the basis of formula (34) we can state the second main result in this section, concerning behaviour of the signature S^-_n for $n = 2m + 1$.

Theorem 18. *Let* $n = 2m + 1$ *and a be the odd sequence. Then for the signatures* S^-_{2m+1} *and* S^-_{2m-1} *there holds the following formula*:

$$S^-_{2m+1} = (s_m + 2) S^-_{2m-1} + (-1)^{m+1} \sum_{i=1}^{m} \omega_i \prod_{j=i+1}^{m} s_j, \tag{35}$$

where $\omega_i = 1 + \prod_{j=1}^{i} s_j \prod_{j=1}^{i-1} (s_j + 2)$ *for* $i = 1, 2, \ldots, m$. *Moreover, for* S_{2m+1}^- *there holds the following identity*:

$$S_{2m+1}^- = Q_m((-1)^{m+1} + \Gamma_m), \tag{36}$$

where $\Gamma_m = (S_{2m-1}^-/Q_m)(s_m + 2)$ *and* $Q_m = \sum_{i=1}^{m} \omega_i \prod_{j=i+1}^{m} s_j$.

Proof. According to Lemma 17 with

$$\omega_i = 1 + (s_1 \cdots s_i)(s_1 + 2) \cdots (s_{i-1} + 2)$$

we obtain

$$\begin{aligned} S_{2m+1}^- &= \sum_{i=1}^{m} \omega_i ((s_i + 2) \cdots (s_m + 2) - s_{i+1} \cdots s_{m+1}) \\ &= \sum_{i=1}^{m-1} \omega_i ((s_i + 2) \cdots (s_m + 2) - q_i + q_i - s_{i+1} \cdots s_{m+1}) + \omega_m ((s_m + 2) - s_{m+1}), \end{aligned}$$

where $q_i \equiv (s_{i+1} \cdots s_m)(s_m + 2)$. Hence, by Lemma 17, we have

$$\begin{aligned} S_{2m+1}^- = (s_m + 2) &\sum_{i=1}^{m-1} \omega_i ((s_i + 2) \cdots (s_{m-1} + 2) - s_{i+1} \cdots s_m) \\ &+ \sum_{i=1}^{m-1} \omega_i (s_{i+1} \cdots s_m)((s_m + 2) - s_{m+1}) + \omega_m ((s_m + 2) - s_{m+1}). \end{aligned}$$

Collecting the last terms, applying identity (17) and by the equality $(s_m + 2) - s_{m+1} = (-1)^{m+1}$, formula (35) follows.
Formula (36) is a consequence of formula (35) and the assumed notation. □

We will prove now that the sign of the signatures S_{2m}^- and S_{2m-1}^- varies periodically as a function of m. The fact that we will know in advance the behaviour of these signatures has a great importance. It allows, namely, to simplify the algorithm *GA*, since we will not have to calculate the signatures in the second step of the algorithm. This implies, in particular, that sequences described by formulae (26) and (27) are output sequences generated by the algorithm *GA* for deterioration rates which are consecutive natural numbers.

Theorem 19. *Let there be given V-sequences* (26) *and* (27) *of the sequence* $a = (1, 2, \ldots, n)$. *Then for each integer* $m \geqslant 1$ *the sign of the signatures* S_{2m}^- *and* S_{2m-1}^- *for these sequences varies periodically according to the formulae*

$$\operatorname{sign}(S_{2m}^-) = (-1)^m \quad \textit{and} \quad \operatorname{sign}(S_{2m-1}^-) = (-1)^m,$$

respectively.

Before we start the proof of Theorem 19, we will prove a few technical lemmas. The main aim of these auxiliary results is to show that quantities R_m, Θ_m, Q_m and Γ_m, which occur in identities (33) and (36), are bounded from above by 1.

Lemma 20. *For each integer* $m \geqslant 1$ *there hold the following recurrence relations*:

$$\Theta_1 = -\tfrac{4}{5}, \quad \Theta_{m+1} = D_m(\Theta_m - (-1)^m) \quad \textit{for } n = 2m \tag{37}$$

and

$$\Gamma_1 = 0, \quad \Gamma_2 = \tfrac{8}{11}, \quad \Gamma_{m+1} = E_m(\Gamma_m - (-1)^m) \quad \textit{for } n = 2m - 1, \tag{38}$$

where

$$D_m = (r_{m+1} + 2) \frac{R_m}{R_{m+1}}$$

and

$$E_m = (s_{m+1} + 2)\frac{Q_m}{Q_{m+1}}.$$

Proof. Recurrence relations (37) and (38) follow immediately from Theorems 16 and 18, respectively. In the case of formula for D_m we additionally apply the equality $r_{m+2} + (-1)^{m+2} = r_{m+1} + 2$. In both formulae, for D_m and for E_m, it is sufficient to apply definitions of Θ_{m+1} and Γ_{m+1}, and the recurrence formulae (30) and (34) for S^-_{2m} and S^-_{2m-1}, respectively. Clearly, the definitions of R_m (cf. Theorem 16) and Q_m (cf. Theorem 18) must be also applied. □

Remark 21. Note that we have $|\Theta_1| < 1$, $\Gamma_1 = 0$ and $|\Gamma_2| < 1$. Moreover, $\Theta_1 < 0$ whereas $\Gamma_2 > 0$.

The next two lemmas are crucial for proofs of inequalities $0 < D_m < 1$ and $0 < E_m < 1$.

Lemma 22. *For each integer $m \geqslant 1$ there holds the following inequality*:

$$R_m < \tfrac{1}{2}\eta_{m+2}.$$

Proof. We will proceed by induction. The case $m=1$ is immediate. Note only that $R_1 = \eta_1 r_1 + \eta_2 = r_1 + (1 + r_1(r_1 - 1))$ and $\eta_3 = 1 + r_1 r_2(r_1 + 2)$, where $r_1 = 2$ and $r_2 = 3$.

Now, assume that $R_{m-1} < \frac{1}{2}\eta_{m+1}$. Hence

$$R_m = r_m R_{m-1} + \eta_{m+1} < \tfrac{1}{2}(r_m + 2)\eta_{m+1}.$$

Thus, it is sufficient to prove that $(r_m + 2)\eta_{m+1} < \eta_{m+2}$. To get this, note first that $(r_m + 2)\eta_{m+1} = (r_m + 2) + (1/r_{m+1})(\eta_{m+2} - 1)$. Consequently, we have to verify that $(r_m + 1) + (1 - (1/r_{m+1})) < (1 - (1/r_{m+1}))\eta_{m+2}$, or that

$$r_m + 1 < \left(1 - \frac{1}{r_{m+1}}\right)(\eta_{m+2} - 1) = \frac{r_{m+1} - 1}{r_{m+1}}(r_1 \cdots r_m r_{m+1})(r_1 + 2)\cdots(r_m + 2).$$

Since $r_{m+1} = r_m + 2 + (-1)^m$, it is sufficient to verify the latter inequality in the expression

$$\frac{r_m + 1}{r_m + 1 + (-1)^m} \leqslant 1 + \frac{1}{r_m} < (r_1 \cdots r_m)(r_1 + 2)\cdots(r_m + 2).$$

Finally, since r_m increases, it is sufficient to verify the case $m = 1$ which is evident. □

Lemma 23. *For each integer $m \geqslant 1$ there holds the following inequality*:

$$Q_m < \tfrac{1}{2}\omega_{m+1}.$$

Proof. First we prove the inequality

$$(s_m + 2)\omega_m < \omega_{m+1}. \tag{39}$$

We have

$$\begin{aligned}(s_m + 2)\omega_m &= (s_m + 2)(1 + (s_1 \cdots s_m)(s_1 + 2)\cdots(s_{m-1} + 2)) \\ &= (s_m + 2) + \frac{1}{s_{m+1}}((1 + (s_1 \cdots s_{m+1})(s_1 + 2)\cdots(s_m + 2)) - 1) \\ &= (s_m + 2) + \frac{1}{s_{m+1}}(\omega_{m+1} - 1).\end{aligned}$$

It is sufficient to verify that $(s_m + 2) + (1/s_{m+1})(\omega_{m+1} - 1) < \omega_{m+1}$ or that

$$\frac{s_m + 1}{s_m + (1 + (-1)^m)} \leqslant 1 + \frac{1}{s_m} < (s_1 \cdots s_m)(s_1 + 2)\cdots(s_m + 2). \tag{40}$$

To get these inequalities we have applied the equality

$$s_{m+1} - 1 = s_m + (1 + (-1)^m).$$

Since $s_i \geqslant 1$, inequalities (40) are trivially satisfied, which completes the proof of inequality (39).

To prove the lemma we will proceed by induction. Let $m = 1$. Since

$$2Q_1 = 2\omega_1 = 2(1 + s_1)$$

and

$$\omega_2 = 1 + (s_1 s_2)(s_1 + 2),$$

we obtain

$$2Q_1 < \omega_2,$$

since $s_1 = 1$ and $s_2 = 2$. Now, assume that holds the inequality $Q_{m-1} < \frac{1}{2}\omega_m$. Since

$$Q_m = \sum_{i=1}^{m} \omega_i s_{i+1} \cdots s_m = s_m Q_{m-1} + \omega_m,$$

we obtain $Q_m < \frac{1}{2}(s_m + 2)\omega_m$. Now, applying inequality (39), we obtain

$$Q_m < \tfrac{1}{2}(s_m + 2)\omega_m < \tfrac{1}{2}\omega_{m+1}$$

as desired. □

Knowing the given above bounds on R_m and Q_m we can prove the following:

Lemma 24. *For each integer $m \geqslant 1$ there hold the following inequalities*:

$$0 < D_m < 1 \quad \textit{and} \quad 0 < E_m < 1.$$

Proof. Clearly, $D_m > 0$ and $E_m > 0$. From definition of D_m, the inequality $D_m < 1$ is equivalent to $R_m < \frac{1}{2}\eta_{m+2}$ which is satisfied in virtue of Lemma 22. To prove that $E_m < 1$, we apply Lemma 23. Indeed, in view of definition of E_m, the inequality $E_m < 1$ is equivalent to the inequality $Q_m < \frac{1}{2}\omega_{m+1}$ from Lemma 23 which completes the proof. □

The next two lemmas show that the sign of quantities Θ_m and Γ_m vary periodically.

Lemma 25. *For each integer $m \geqslant 1$ there holds the inequality $|\Theta_m| < 1$ and the equality* $\mathrm{sign}(\Theta_m) = (-1)^m$.

Proof. Taking $m = 2k$ or $m = 2k - 1$, for each integer $k \geqslant 1$ we obtain, by Lemma 20, respectively:

$$\Theta_{2k+1} = D_{2k}(\Theta_{2k} - (-1)^{2k}) = D_{2k}(\Theta_{2k} - 1)$$

and

$$\Theta_{2k} = D_{2k-1}(\Theta_{2k-1} - (-1)^{2k-1}) = D_{2k-1}(\Theta_{2k-1} + 1).$$

After substitution for odd indices the value $2k - 1$ we have

$$\Theta_{2k+1} = D_{2k}(D_{2k-1}(\Theta_{2k-1} + 1) - 1).$$

To prove that $|\Theta_{2k-1}| < 1$ and $\Theta_{2k-1} < 0$ for each integer $k \geqslant 1$ we will proceed by induction.

For $k = 1$ we have $|\Theta_1| < 1$ and $\Theta_1 < 0$, since $\Theta_1 = -\frac{4}{5}$ by definition. Assume that $|\Theta_{2k-1}| < 1$ and $\Theta_{2k-1} < 0$. We will prove that $|\Theta_{2k+1}| < 1$ and $\Theta_{2k+1} < 0$. By induction assumption $0 < D_{2k-1}(\Theta_{2k-1} + 1) < 1$ and, consequently, $0 < 1 - D_{2k-1}(\Theta_{2k-1} + 1) < 1$. Hence $|\Theta_{2k+1}| < 1$ and $\Theta_{2k+1} < 0$ as desired.

Now, consider the case of even indices, $m = 2k$. Applying the odd case we obtain

$$\Theta_{2k} = D_{2k-1}(\Theta_{2k-1} + 1) > 0$$

with $D_{2k-1}(\Theta_{2k-1} + 1) < 1$. Consequently, $|\Theta_{2k}| < 1$ with $Q_{2k} > 0$. Collecting this all together we get our thesis and the proof is completed. $\square$

Lemma 26. *For each integer $m > 1$ there holds the inequality $|\Gamma_m| < 1$ and the equality* $\operatorname{sign}(\Gamma_m) = (-1)^m$.

Proof. Taking $m = 2k$ or $m = 2k - 1$, for each integer $k \geqslant 1$ we obtain, by Lemma 20, respectively:

$$\Gamma_{2k+1} = E_{2k}(\Gamma_{2k} - (-1)^{2k}) = E_{2k}(\Gamma_{2k} - 1)$$

and

$$\Gamma_{2k} = E_{2k-1}(\Gamma_{2k-1} - (-1)^{2k-1}) = E_{2k-1}(\Gamma_{2k-1} + 1).$$

After substitution for odd indices the value $2k - 1$ we have

$$\Gamma_{2k+1} = E_{2k}(E_{2k-1}(\Gamma_{2k-1} + 1) - 1).$$

To prove that $|\Gamma_{2k-1}| < 1$ and $\Gamma_{2k-1} < 0$ for each integer $k \geqslant 2$ we will proceed by induction.

Note that for $k = 1$ we have $\Gamma_1 = 0$. For $k = 2$ there holds $|\Gamma_3| < 1$ and $\Gamma_3 < 0$, since $\Gamma_3 = E_2(\Gamma_2 - 1) = E_2(\frac{8}{11} - 1)$ and $0 < E_m < 1$.

Now, let $|\Gamma_{2k-1}| < 1$ and $\Gamma_{2k-1} < 0$ for arbitrary $k > 2$. Then

$$-1 < E_{2k-1}(\Gamma_{2k-1} + 1) - 1 < 0,$$

since $0 < E_{2k-1}(\Gamma_{2k-1} + 1) < 1$. Finally, we obtain $|\Gamma_{2k+1}| < 1$ and $\Gamma_{2k+1} < 0$, what finishes the induction step. This result implies that

$$\Gamma_{2k} = E_{2k-1}(\Gamma_{2k-1} + 1) > 0 \quad \text{and} \quad \Gamma_{2k} < 1$$

for each integer $k \geqslant 2$. Moreover, $\Gamma_2 = \frac{8}{11}$, i.e. $\Gamma_2 > 0$ and $\Gamma_2 < 1$. This completes the proof. $\square$

Finally, Lemmas 25 and 26 allow us to prove Theorem 19.

Proof of Theorem 19. Indeed, in virtue of formula

$$S^-_{2m+2} = R_m((-1)^{m+1} + \Theta_m)$$

for each integer $m \geqslant 2$, the fact that $\operatorname{sign}(S^-_2) = 1$ and applying Lemma 25, we conclude that $\operatorname{sign}(S^-_{2m}) = (-1)^m$ for each integer $m \geqslant 1$.

Similarly, in virtue of formula

$$S^-_{2m+1} = Q_m((-1)^{m+1} + \Gamma_m)$$

for each integer $m \geqslant 1$, the fact that $\operatorname{sign}(S^-_1) = -1$ and applying Lemma 26, we get that the equality $\operatorname{sign}(S^-_{2m-1}) = (-1)^m$ holds for each integer $m \geqslant 1$. This completes the proof. $\square$

Summarize now the results of this section. First, we have shown that for deterioration rates which are consecutive natural numbers the sign of the signature $S^-(\cdot)$ varies periodically. This implies, in particular, that for the sequence $a = (1, 2, \ldots, n)$ the greedy algorithm *GA* generates an approximate solution of the $F(a)$-problem which is a V-sequence of the form of (26) or (27). Second, computational experiment carried out for $n = 2, 3, \ldots, 20$ has shown that solutions of the $F(a)$-problem with these V-shaped sequences are equivalent to optimal solutions of the $F(a)$-problem for both, even and odd sequences. These results lead us to the following:

Conjecture 27. *The algorithm GA is optimal for the TDMTCT problem in the case when* $\alpha_j = j$ *for* $j = 0, 1, \ldots, n$.

Finally, note that solutions of the $F(a)$-problem for $a = (1, 2, \ldots, n)$ can be found in $\mathrm{O}(n)$ time since, by Theorem 19, we do not have to calculate signatures at the second step of the algorithm *GA*. Moreover, similar fact will hold for any sequence a for which a theorem analogical to Theorem 19 will be proved.

7. Conclusions

In the paper we considered a single machine time-dependent scheduling problem. First, we reformulated the initial problem as a matrix equation over the set of all permutations of a given sequence of job deterioration rates. Second, we defined two l_p-norm-related measures, we introduced the new notion of signatures of a sequence and proved some of their properties.

Using the results and notions mentioned above, we formulated a greedy algorithm for the problem. The algorithm, starting from an initial sequence, iteratively constructs a new sequence concatenating the previously obtained sequence with new elements, according to the sign of one of signatures of this sequence.

We proved that for job deterioration rates which are consecutive natural numbers the signatures of sequences generated by the greedy algorithm vary periodically as consecutive powers of (-1). Knowing this, we were able to construct these sequences explicitly. Moreover, conducted computational experiment allowed us to formulate the conjecture that in this case the proposed algorithm is optimal.

We think that the approach presented in the paper can also be applied to other types of sequences, e.g. arithmetic or geometric ones. This topic is currently subject of our research.

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